1 Fundamentals and Techniques

1.1 Complex Numbers and Elementary Functions

1.1.1 Properties

We define an imaginary number as

$$i^2 = -1$$

While a complex number is defined as

$$z = x + iy$$

The common functions $\Re$ and $\Im$ yield the real and imaginary parts of a complex number respectively.[1] We can also express complex numbers in polar coordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Using Euler's Identity,

$$\cos\theta + i\sin\theta = e^{i\theta}$$

the alternate form is defined as

$$z = x + iy = r\left(\cos\theta + i\sin\theta\right) = re^{i\theta}$$
$$r = \sqrt{x^2+y^2} = |z|$$
$$\tan\theta = \frac{y}{x}$$

The complex conjugate is defined as

$$x - iy \equiv re^{-i\theta}$$

We can define some common equivalences.

- $\exp(2\pi i) = 1$
- $\exp(\pi i) = -1$
- $\exp\left(\frac{\pi i}{2}\right) = i$
- $\exp\left(\frac{3\pi i}{2}\right) = -i$
- $\exp(i\theta_1)\exp(i\theta_2) = \exp\left(i\left(\theta_1+\theta_2\right)\right)$
- $\exp(i\theta)^m = \exp(im\theta)$
- $\exp(i\theta)^{1/n} = \exp\left(\frac{i\theta}{n}\right)$

Another neat trick is to let $z = 1/t$ to analyze behavior at ∞.

[1] Also denoted as Re and Im

1.1.2 Stereographic Projection

We can visualize complex numbers with a stereographic projection. Zero is located at the North Pole, and infinity at the South Pole.

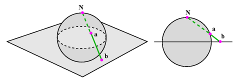

Figure 1: Stereographic Projection

These points are

$$X = \frac{4x}{|z|^2+4} \qquad Y = \frac{4y}{|z|^2+4} \qquad Z = \frac{2|z|^2}{|z|^2+4}$$

1.1.3 Elementary Functions

Similar to Real Analysis, we can define a neighborhood of some point z as the region enclosed by

$$|z - z_0| < \epsilon$$

As with sets, these can be closed, bounded, regions, domains, etc. . . .

We can also define functions of complex numbers, and as with real valued numbers, they mostly work the same. The simplest function is the power function.

$$f(z) = z^n$$

Which can be extended to define Polynomials and rational functions (as the result of dividing a polynomial function with another).

Limits also work the same, even with Radii of Convergence, etc.

Projections and Mappings work intuitively.

1.1.4 Example

Solve for all roots of the following equation: $z^4 + 2z = 0$.

$z\left(z^3+2\right) = 0$, so $z = 0$ or $z^3 = -2$, and then $r^3 = 2$, $e^{3i\theta} = e^{i\pi} \Rightarrow \theta = \pi/3 + 2\pi n/3$, $n = 0, 1, 2$. Thus, the roots are

$$z = 0, 2^{1/3}e^{i\pi/3}, 2^{1/3}e^{i\pi} = -2^{1/3}, 2^{1/3}e^{5i\pi/3}$$

1.1.5 Limits

Theorem 1 ($\epsilon - \delta$ Limit Definition). *A complex limit can be defined as*

$$\lim_{z\to z_0} f(z) = w_0$$

if for every sufficiently small $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \qquad |z - z_0| < \delta$$

This is the traditional $\epsilon - \delta$ format that we're used to from real analysis.

Similarly, a function is said to be continuous if for all z,

$$\lim_{z\to z_0} f(z) = z_0$$

The traditional definitions of Uniform and Absolute convergence also apply.

Using these limit definitions we can define the concept of a derivative.

$$f'(z_0) = \lim_{\Delta z\to 0}\left(\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}\right) = \lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)$$

1.2 Analytic Functions and Integration

1.2.1 Analytic Functions

In order for a complex function to be differentiable, it has to satisfy the Cauchy-Riemann Conditions.

Theorem 2 (Cauchy-Riemann Conditions). *By writing the real and imaginary parts separately in the definition of a derivative, we get*

$$f(z) = u(x,y) + iv(x,y)$$
$$f'(z) = \lim_{\Delta x\to 0}\left(\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x} + i\frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}\right)$$
$$= u_x(x,y) + iv_x(x,y)$$

Yielding the Cauchy-Riemann conditions,

$$u_x = v_y \qquad v_x = -u_y$$
$$u_r = \frac{v_\theta}{r} \qquad v_r = -\frac{u_\theta}{r}$$

Theorem 3. *The function $f(z) = u(x,y) + iv(x,y)$ is differentiable at a point $z = x + iy$ of a region in the complex plane if and only if the partial derivatives u_x, u_y, v_x, v_y, are continuous and satisfy the Cauchy-Riemann conditions at $z = x + iy$.*

For differentiability, we can use the term analyticity to mean the same thing, both for pointwise differentiability and differentiability over a region. Points that are not differentiable (analytic) are called singular points.[2]

Some properties follow.[3]

- Sums, Products, and Compositions of analytic functions are analytic.
- The reciprocal of an analytic function that is nowhere zero is analytic, as is the inverse of an invertible analytic function whose derivative is nowhere zero.

An entire function is one that's analytic on the entire finite plane.

Taking the second derivative of the Cauchy-Riemann conditions yields Laplace's Equation.

$$u_{xx} = v_{xy} \qquad v_{yx} = -u_{yy}$$

$$\nabla^2 w = 0 \Rightarrow \begin{cases}\nabla^2 u \equiv u_{xx} + u_{yy} = 0\\ \nabla^2 v \equiv v_{xx} + v_{yy} = 0\end{cases}$$

A function that satisfies the concise Laplace Equation: $\nabla^2 w = 0$ is called a harmonic function in D. u and v are referred to as harmonic functions in D, and they are harmonic conjugates of each other.

1.2.2 Example

Let $f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x\left(\cos y + i\sin y\right)$. Verify Cauchy-Riemann for all x, y, and then show that $f'(z) = e^z$.

$$u = e^x\cos y \qquad v = e^x\sin y$$
$$u_x = e^x\cos y = v_y$$
$$u_y = -e^x\sin y = -v_x$$
$$f'(z) = u_x + iv_x = e^x\left(\cos y + i\sin y\right) = e^z$$

1.2.3 Ideal Fluid Flow - Application of Laplace's Equation

Two dimensional ideal fluid flow is a great example of Laplace's Equation. This is fluid that is time independent, nonviscous, incompressible, and irrotational.

1. Incompressibility:

$$v_{1,x} + v_{2,y} = 0$$

[2] Holomorphic is sometimes used as well (or instead) of analytic.

[3] https://en.wikipedia.org/wiki/Analytic_function#Properties_of_analytic_functions

Where v_1 and v_2 are the horizontal and vertical components.

2. Irrotationality:

$$v_{2,x} - v_{1,y} = 0$$

3. Simplified:

$$v_1 = \phi_x = \psi_y \qquad v_2 = \phi_y = -\psi_x$$
$$\mathbf{v} = \nabla\phi$$

ϕ is the velocity potential, and ψ the stream function. Cauchy-Riemann is satisfied for ϕ and ψ, therefore we have a complex velocity potential.

$$\Omega(z) = \phi(x,y) + i\psi(x,y)$$
$$\Omega'(z) = \phi_x + i\psi_x = \phi_x - i\psi_y = v_1 - v_2$$

1.2.4 Example

Uniform Flow is

$$\Omega(z) = v_0 e^{-i\theta_0} z = v_0\left(\cos\theta_0 - i\sin\theta_0\right)\left(x + iy\right)$$

where v_0 and θ_0 are positive real constants. The corresponding velocity potential and velocity field is given by

$$\phi(x,y) = v_0\left(\cos\left(\theta_0 x\right) + \sin\left(\theta_0 y\right)\right)$$
$$v_1 = \phi_x = v_0\cos\theta_0 \qquad v_2 = \phi_y = v_0\sin\theta_0$$

which is identified with uniform flow making an angle θ_0 with the x axis. Alternatively, the steam function $\psi(x,y) = v_0\left(\cos\left(\theta_0 y\right) - \sin\left(\theta_0 x\right)\right) = \text{const.}$ reveals the same flow field.

1.2.5 Multivalued Functions

A simple example of this is the square root function which takes on different values for n even or odd.

$$z = w^2 \qquad w = \sqrt{z}$$
$$= r^{1/2}e^{i\theta_p/2}e^{n\pi i}$$

We can define these "points" where complex functions take on multiple values as branch points. In the same way that they're referred to as branch points, branches of a multivalued function are when we restrict to only one set of continuous values. A branch cut is this restriction process.[4]

Log is more complicated, and we define it as such.

$$w = \log(z) = \log r + i\theta_p + 2n\pi i, \qquad n = 0, \pm1, \pm2, \ldots, \qquad 0 \le \theta_p < 2\pi$$

1.2.6 Example

Find the location of the branch points and discuss possible branch cuts for the following functions:

1. $(z-i)^{1/3}$
   Let $z - i = \epsilon e^{i\theta_p}$ which is a circular contour centered at $z = i$. We have just a power function in terms of $\zeta = z - i$, so $z = i$ and $z = \infty$ are branch points. Any line connecting $z = \infty$ and $z = i$ is a branch cut, e.g. $\{z = iy | y \in [1, +\infty)\}$ is as good as any. There are 3 distinct branches.
2. $\log\left(\frac{1}{z-2}\right)$
   $\log\left(\frac{1}{z-2}\right) = -\log\left(z-2\right)$. Again, this is $-\log(z)$ but with shiftd origin. So the branch points are $z = 2$ and $z = \infty$. A branch cut must connect the branch points, it can be $\{z = x | x \in [2, +\infty)\}$ or $\{z = x | x \in (-\infty, 2]\}$.

1.2.7 Example

Solve for all values of z: $4 + 2e^{z+i} = 2$.

$$4 + 2e^{z+i} = 2 \Rightarrow e^{z+i} = -1 = e^{i\pi + 2\pi i n}, n \in \mathbb{Z}$$

Therefore

$$z + i = i\pi + 2\pi i n \Rightarrow z = i\left(\pi - 1 + 2\pi n\right), n \in \mathbb{Z}$$

1.2.8 Example

Find the location of the branch points and discuss a branch cut structure associated with the function:

- $f(z) = \frac{z-1}{z}$
  This is a rational function singular at $z = 0$, but single-valued, so no branch points.
- $f(z) = \log\left(z^2 - 3\right)$
  Here $z^2 - 3$ is entire single-valued function so the only branch points are those where $z^2 - 3 = 0$ or $z^2 - 3 = \infty$. Thus, there are three branch points, $z = \pm\sqrt{3}$, and $z = \infty$. A branch cut must make sure there is no possibility going around and single of them, in this case it must connect all three points. E.g. consider a cut on real axis $\{z = x | x \in [-3, +\infty)\}$.
- $f(z) = \exp\sqrt{z^2 - 1}$
  Since function e^z is entire (analytic on plane) the only possible branch points are those of $\sqrt{z^2-1}$, i.e. $z = \pm1$ and $z = \infty$. However, doing the circle argument $z - 1 = r_1e^{i\theta_1}$, $z + 1 = r_2e^{i\theta_2}$, $\theta_1 \to \theta_1 + 2\pi$, $\theta_2 \to \theta_2 + 2\pi$, one sees that $z = \infty$ is not a branch point since $\exp\left(2\pi i + 2\pi i\right)/2 = 1$ which corresponds to encircling both $z = 1$ and $z = -1$, equivalent to encircling just $z = \infty$. Thus, $z = \infty$ is not a branch point even for $\sqrt{z^2-1}$. But $z = \pm1$ are branch points, and a branch cut connecting them is $\{z = x | x \in [-1, 1]\}$.

[4] The real analogy here is a function like $\pm\sqrt{x}, x \in \mathbb{R}$. 0 is a branch point, and we often times just examine the branch where $\sqrt{x} > 0$. The analogous branch cut is $x > 0$.

1.2.9 More Complicated Multivalued Functions and Riemann Surfaces

If we have functions like the following

$$w = \left[\left(z-a\right)\left(z-b\right)\right]^{1/2}$$

We need to use a slightly more complicated branch cut/structure. We know that the points $z = a, b$ are both branch points (by letting $z = a + \epsilon_1 e^{i\theta_1}$ and as θ_1 varies from 0 to 2π, w jumps from $q^{1/2}$ to $-q^{1/2}$), and so we can define a branch cut as follows.

$$z - b = r_1e^{i\theta_1}$$
$$z - a = r_2e^{i\theta_2} \qquad 0 \le \theta_1, \theta_2 < 2\pi$$

Our equation now becomes

$$w = \left(r_1r_2\right)^{1/2}e^{i(\theta_1+\theta_2)/2}$$

This process extends to more complicated functions, as for any w of the form

$$w = \left[\left(z-x_1\right)\left(z-x_2\right)\cdots\left(z-x_n\right)\right]^m$$

we can define our branch cuts to be

$$z - x_k = r_ke^{i\theta_k}$$

yielding

$$w = \left(r_1r_2\cdots r_n\right)e^{mi(\theta_1+\theta_2+\cdots+\theta_n)}$$

1.2.10 Example

Find the location of branch points and discuss a branch cut structure associated with the function:

$$f(z) = \coth^{-1}\frac{z}{a} = \frac{1}{2}\log\left(\frac{z+a}{z-a}\right), a > 0$$

This is (up to a constant) log of rational function, so the branch points are those where $\left(z+a\right)/\left(z-a\right) = 0$ or ∞, i.e. $z = \pm a$. As for $z = \infty$, it is not a branch point, as the limit equals 1, not zero. A cut must connect the two points, so a possible one is interval $[-a, a]$ on the real axis.

1.2.11 Riemann Surfaces

Instead of considering the normal complex plane with arbitrary "cuts", it can be useful to instead consider a surface with multiple "sheets". Any multivalued function only has one point that corresponds to each point on the sheet. This way, for any given sheet, the function is single-valued.

For the function $w^{1/2}$, since we have two branches, our Riemann surface is two-sheeted. For the log function, since it is infinitely multivalued, we have infinite sheets.

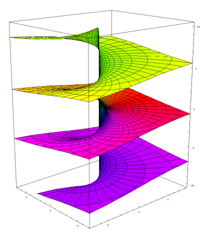

Figure 2: Riemann Surface for $\log(z)$

1.2.12 Complex Integration

Consider a function $f(t) = u(t) + iv(t)$. This function is integrable if u and v are integrable (with the same properties applying).

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Defining a curve on the complex plane can be done parametrically, with form

$$z(t) = x(t) + iy(t)$$

The path (contour) integral of function f on contour z is defined to be

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$$

This is really a line integral in the (x, y) plane.

Theorem 4. *Suppose $F(z)$ is an analytic function and that $f(z) = F'(z)$ is continuous in a domain D. Then for a contour C lying in D with endpoints z_1 and z_2*

$$\int_C f(z)\,dz = F(z_2) - F(z_1)$$

For closed curves, we have

$$\oint_C f(z)\,dz = \oint_C F'(z)\,dz = 0$$

Note that everything here hinges on the analyticity of F and the continuity in domain D.

Theorem 5. *Let $f(z)$ be continuous on a contour C. Then*

$$\left|\int_C f(z)\,dz\right| \le ML$$

where L is the length of C and M is an upper bound for $|f|$ on C.

Arc length can be defined (from Calc III) for a parameterized curve with form $z(t) = u(t) + iv(t)$ as

$$\int_a^b \sqrt{\left(u'(t)\right)^2 + \left(v'(t)\right)^2}\,dt$$

1.2.13 Example

Evaluate $\int_C \overline{z}\,dz$ for a contour from $z = 0$ to $z = 1$ to $z = 1 + i$.

$$\int_C \overline{z}\,dz = \int_C \left(x - iy\right)\left(dx + i\,dy\right)$$
$$= \int_{x=0}^1 x\,dx + \int_{y=0}^1 \left(1 - iy\right)\left(i\,dy\right)$$
$$= \frac{1}{2} + i\left[y - iy^2/2\right]_0^1$$
$$= 1 + i$$

1.2.14 Cauchy's Theorem

Theorem 6 (Cauchy). *If a function f is analytic in a simply connected domain D, then along a simple closed contour C in D*

$$\oint_C f(z)\,dz = 0$$

Again, NOTE that everything hinges on the fact that D must be simply connected. In order to use this, you need a simply connected domain D AND a simple closed contour C.

To best apply Cauchy's Theorem, we can use tricks like turning a complex contour into several simple contours, and deforming a simply connected domain so that the function is analytic on the domain.